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# Similarity solutions of the Euler equation in the calculus of variations

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**Abstract.** A necessary and sufficient condition for the equivalence of the conformal invariance of the Euler equation and the divergence invariance of the associated variational integral is obtained in terms of the generators of the local Lie group of transformations under which invariance is studied.

## 1. Introduction

In this paper we explore the connection between the divergence invariance of the fundamental integral in the calculus of variations and the conformal invariance of the corresponding Euler equation, the latter of which would give rise to similarity solutions; the invariance is with respect to a one-parameter, local Lie group of transformations. In particular, we consider integral functionals of the form

$$J(x) := \int_{t_0}^{t_1} L(t, x, \dot{x})\, \mathrm{d}t, \qquad \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}, \tag{1.1}$$

where $x \in C^2[t_0, t_1]$ and the Lagrange function $L$ is four times continuously differentiable in $[t_0, t_1] \times R^2$. The corresponding Euler equation is

$$E := L_x - L_{\dot{x}t} - L_{\dot{x}x}\dot{x} - L_{\dot{x}\dot{x}}\ddot{x} = 0, \tag{1.2}$$

where subscripts denote partial differentiation. We also consider a one-parameter local Lie group of transformations which we write in infinitesimal form as

$$\bar{t} = t + \varepsilon\tau(t) + \mathrm{o}(\varepsilon), \qquad \bar{x} = x + \varepsilon\xi(x, t) + \mathrm{o}(\varepsilon). \tag{1.3}$$

Here, $\varepsilon$ is a real parameter which varies over some open interval $I$ containing zero, and $\tau$ and $\xi$ are the group generators defined by

$$\tau(t) = (\partial\bar{t}/\partial\varepsilon)_{\varepsilon=0}, \qquad \xi(t, x) = (\partial\bar{x}/\partial\varepsilon)_{\varepsilon=0},$$

and $\mathrm{o}(\varepsilon)$ denotes terms for which $\mathrm{o}(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \to 0$.

We say that the integral functional (1.1) is divergence invariant under (1.3) if, and only if, there exists a twice continuously differentiable function $\phi(t, x)$ for which

$$L(\bar{t}, \bar{x}, \mathrm{d}\bar{x}/\mathrm{d}\bar{t})\, \mathrm{d}\bar{t}/\mathrm{d}t = L(t, x, \dot{x}) + (\mathrm{d}/\mathrm{d}t)\phi(t, x)\varepsilon + \mathrm{o}(\varepsilon) \tag{1.4}$$

for all $\varepsilon$ in $I$.

If $\phi \equiv 0$, then (1.1) is said to be absolutely invariant (see Logan 1977). It was proved by Rund (1972) that a necessary and sufficient condition for (1.1) to be

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divergence invariant under (1.3) is that the Lagrange function $L$ and the group generators $\tau$ and $\xi$ satisfy the invariance identity

$$L_t\tau + L_x\xi + L_{\dot{x}}(\dot{\xi} - \dot{x}\dot{\tau}) + L\dot{\tau} = \dot{\phi}. \tag{1.5}$$

Here, $\dot{\xi}$ and $\dot{\tau}$ denote the total derivatives of $\xi$ and $\tau$, i.e., $\dot{\xi} = \xi_t + \xi_x\dot{x}$, $\dot{\tau} = \tau_t$, and similarly for $\dot{\phi}$.

To define invariance of a second-order differential equation we require the concept of the extended group which defines how derivatives transform under (1.3). It is well known (see, e.g., Logan 1977) that this extended transformation is given by

$$\mathrm{d}\bar{x}/\mathrm{d}\bar{t} = \dot{x} + \varepsilon(\dot{\xi} - \dot{x}\dot{\tau}) + \mathrm{o}(\varepsilon), \qquad \mathrm{d}^2\bar{x}/\mathrm{d}\bar{t}^2 = \ddot{x} + \varepsilon(\ddot{\xi} - 2\ddot{x}\dot{\tau} - \dot{x}\ddot{\tau}) + \mathrm{o}(\varepsilon), \tag{1.6}$$

coupled with (1.3). The terms

$$p := \dot{\xi} - \dot{x}\dot{\tau}, \qquad q := \ddot{\xi} - 2\ddot{x}\dot{\tau} - \dot{x}\ddot{\tau} \tag{1.7}$$

define the generators. The Lie derivative operator of the extended group will be denoted by

$$\mathscr{L} := \tau\,\partial/\partial t + \xi\,\partial/\partial x + p\,\partial/\partial\dot{x} + q\,\partial/\partial\ddot{x}. \tag{1.8}$$

A second-order differential equation

$$F(t, x, \dot{x}, \ddot{x}) = 0 \tag{1.9}$$

is said to be conformally invariant under (1.3) if, and only if,

$$\mathscr{L}F = \alpha F, \tag{1.10}$$

for some function $\alpha = \alpha(t, x)$ (see Ames (1972) or Bluman and Cole (1974)).

The question addressed in this paper is the relation between divergence invariance of a variation integral (1.1) and the conformal invariance of the Euler equation (1.2).

## 2. A counterexample

It is not difficult to construct an example to show that invariance of the Euler equation does not imply invariance of the corresponding integral functional. Take

$$J(x) = \int_{t_0}^{t_1} (x\dot{x}^3 + x/\sqrt{t})\,\mathrm{d}t, \tag{2.1}$$

whose Euler equation is the nonlinear equation

$$F(t, x, \dot{x}, \ddot{x}) := -6x\dot{x}\ddot{x} - 2\dot{x}^3 + 1/\sqrt{t} = 0. \tag{2.2}$$

Consider the group of stretching transformations

$$\bar{t} = (1+\varepsilon)t, \qquad \bar{x} = (1+\varepsilon)^{5/6}x \tag{2.3}$$

with generators

$$\tau = t, \qquad \xi = \tfrac{5}{6}x.$$

It is easy to see that

$$\mathscr{L}F = -\tfrac{1}{2}F,$$

so that (2.2) is conformally invariant with conformal factor $\alpha = -\frac{1}{2}$. However, a straightforward calculation shows that neither (1.4) nor (1.5) holds for any choice of $\phi$, and therefore the fundamental integral is not invariant.

Two questions arise naturally. Are there supplementary conditions which will guarantee this implication, and is the converse implication true? In the case of the Emden–Fowler equation, which we write as

$$F(t, x, \dot{x}, \ddot{x}) := -t^2\ddot{x} - 2t\dot{x} - t^2x^5 = 0,$$

we find that both the differential equation and the corresponding integral functional

$$J(x) := = \int_{t_0}^{t_1} t^2\left(\frac{\dot{x}^2}{2} - \frac{x^6}{6}\right) \mathrm{d}t$$

are invariant under the group

$$\bar{t} = (1+\varepsilon)t, \qquad \bar{x} = (1-\varepsilon/2)x,$$

with $\mathscr{L}F = -\frac{1}{2}F$.

## 3. Conditions for invariance

In this section we shall obtain a necessary and sufficient condition for these two types of invariance to be equivalent. We shall show that the conformal factor $\alpha$ must be given in terms of the group generators; the divergence term $\phi$ plays no role. In fact, we shall prove the following theorem.

*Theorem.* If $J(x)$ is divergence invariant under the one-parameter group of transformations (1.3), then the Euler equation is conformally invariant under (1.3) with conformal factor

$$\alpha = -(\xi_x + \tau_t). \tag{3.1}$$

The converse also holds.

*Proof.* The proof in the forward direction involves a lengthy, although straightforward, calculation of $\mathscr{L}E$, where $E$ is the Euler expression defined by (1.2), using the invariance identity (1.5). The key to the converse is to consider $L_{\dot{x}\dot{x}}$ rather than the Lagrange function $L$ itself.

First, we have

$$\begin{aligned}\mathscr{L}E = (L_{xt} - L_{\dot{x}tt} - L_{\dot{x}xt}\dot{x} - L_{\dot{x}\dot{x}t}\ddot{x})\tau + (L_{xx} - L_{\dot{x}tx} - L_{\dot{x}xx}\dot{x} - L_{\dot{x}\dot{x}x}\ddot{x})\xi \\ - (L_{\dot{x}\dot{x}t} + L_{\dot{x}\dot{x}x}\dot{x} + L_{\dot{x}\dot{x}\dot{x}}\ddot{x})(\dot{\xi} - \dot{x}\tau_t) - L_{\dot{x}\dot{x}}(\ddot{\xi} - 2\ddot{x}\tau_t - \dot{x}\tau_{tt}).\end{aligned} \tag{3.2}$$

We show that $\mathscr{L}E = \alpha E$ for some $\alpha$. Noting that (1.5) is an *identity* for arbitrary directions $\dot{x}$, we can differentiate (1.5) with respect to $\dot{x}$ to obtain

$$L_{t\dot{x}}\tau + L_{x\dot{x}}\xi + L_{\dot{x}}(\xi_x - \tau_t) + L_{\dot{x}\dot{x}}(\dot{\xi} - \dot{x}\tau_t) + \tau_t L_{\dot{x}} = \phi_x. \tag{3.3}$$

Differentiating (3.3) with respect to $t$, $x$, and $\dot{x}$ gives the three identities

$$\begin{aligned}L_{tt\dot{x}}\tau + L_{x\dot{x}t}\xi + L_{\dot{x}\dot{x}t}(\dot{\xi} - \dot{x}\tau_t) = -[L_{t\dot{x}}\tau_t + L_{x\dot{x}}\xi_t + L_{\dot{x}t}(\xi_x - \tau_t) \\ + L_{\dot{x}}(\xi_{xt} - \tau_{tt}) + L_{\dot{x}\dot{x}}(\xi_{tt} + \dot{x}\xi_{xt} - \dot{x}\tau_{tt}) + \tau_t L_{\dot{x}t} + \tau_{tt}L_{\dot{x}}] + \phi_{xt}\end{aligned} \tag{3.4}$$

$$L_{xt\dot{x}}\tau + L_{x\dot{x}x}\xi + L_{\dot{x}\dot{x}x}(\dot{\xi} - \dot{x}\tau_t) = -[\xi_x L_{\dot{x}x} + \xi_{xx} L_{\dot{x}} + L_{\dot{x}x}(\xi_x - \tau_t)$$
$$+ L_{\dot{x}\dot{x}}(\xi_{tx} + \dot{x}\xi_{xx}) + \tau_t L_{\dot{x}x}] + \phi_{xx} \tag{3.5}$$
$$L_{\dot{x}\dot{x}t}\tau + L_{\dot{x}\dot{x}t}\xi + L_{\dot{x}\dot{x}\dot{x}}(\dot{\xi} - \dot{x}\tau_t) = -(2\xi_x - \tau_t)L_{\dot{x}\dot{x}}. \tag{3.6}$$

Finally, differentiation of (1.5) with respect to $x$ gives

$$L_{tx}\tau + L_{xx}\xi = -[L_x\xi_x - L_{\dot{x}x}(\dot{\xi} - \dot{x}\tau_t) + L_{\dot{x}}(\xi_{tx} - \dot{x}\xi_{xx}) + L_x\tau_t] + \phi_{tx} + \dot{x}\phi_{xx}. \tag{3.7}$$

Substituting (3.4) through (3.7) into (3.2) gives, after considerable reduction,

$$\mathscr{L}E = -(\xi_x + \tau_t)E. \tag{3.8}$$

To prove the converse, we notice (and this is the key to the argument) that (1.10), and hence (3.8), is an identity in the independent variables $t$, $x$, $\dot{x}$, and $\ddot{x}$. By the calculation, it is seen that $\ddot{x}$ occurs linearly in (3.8), and therefore its coefficient is zero. That is,

$$L_{\dot{x}\dot{x}t}\tau + L_{\dot{x}\dot{x}x}\xi + L_{\dot{x}\dot{x}\dot{x}}(\dot{\xi} - \dot{x}\tau_t) + L_{\dot{x}\dot{x}}(2\xi_x - \tau_t) = 0. \tag{3.9}$$

But this is precisely equation (3.6) which is a twice differentiated form (with respect to $\dot{x}$) of the invariance identity (1.5). Therefore, we integrate (3.9) twice, partially with respect to $\dot{x}$ to get

$$L_t\tau + L_x\xi + L_{\dot{x}}(\dot{\xi} - \dot{x}\tau_t) + L\tau_t = a(x, t) + b(x, t)\dot{x} \tag{3.10}$$

for some functions $a(x, t)$ and $b(x, t)$. It remains to show that the right-hand side of (3.10) is a total derivative.

Equation (3.10) can be differentiated to obtain the identities (3.4) through (3.7) with $\phi_{tx}$, $\phi_{xx}$, and $\phi_{xt}$ replaced by $a_x$, $b_x$, and $b_t$, respectively. Substituting those identities into the expression $\mathscr{L}E$ given by (3.2), we obtain

$$\mathscr{L}E = -(\xi_x + \tau_t)E + a_x - b_t.$$

Hence, $a_x = b_t$, and therefore there exists a function $\psi\ (t, x)$ for which

$$a + b\dot{x} = \dot{\psi}.$$

Therefore the theorem is proved.

This explains the examples in § 2. The Emden-Fowler equation satisfies the conformality condition

$$a = -(\tau_t + \xi_x) = -\tfrac{1}{2}.$$

However, for (2.1), (2.2) we have $\alpha = -\frac{1}{2}$, whereas $-(\tau_t + \xi_x) = -\frac{11}{6}$.

## 4. Generalisation to vector functions

When $x = (x^1, \ldots, x^n)$ and the transformation is given by

$$\bar{t} = t + \varepsilon\tau(t) + o(\varepsilon)$$
$$\bar{x}^k = x^k + \varepsilon\xi^k(t, x) + o(\varepsilon), \qquad k = 1, \ldots, n \tag{4.1}$$

then the invariance identity (1.5) becomes

$$L_t\tau + L_{x^k}\xi^k + L_{\dot{x}^k}(\dot{\xi}^k - \dot{x}^k\tau_t) + L\tau_t = \mathrm{d}\phi(t, x)/\mathrm{d}t,$$

where the summation convention is assumed.

The Euler expressions are

$$E_k := L_{x^k} - L_{\dot{x}^k t} - L_{\dot{x}^k x^l}\dot{x}^l - L_{\dot{x}^k \dot{x}^l}\ddot{x}^l$$

for $k = 1, \ldots, n$. In this case, we say that the system of Euler equations $E_k = 0$, $k = 1, \ldots, n$ is conformally invariant under (4.1) if, and only if,

$$\mathscr{L}E_k = \alpha^l_k E_l, \qquad k = 1, \ldots, n$$

for some functions $\alpha^l_k = \alpha^l_k(t, x)$, $k$, $l = 1, \ldots, n$. Here,

$$\mathscr{L} = \tau\, \partial/\partial t + \xi^l\, \partial/\partial x^l + p^l\, \partial/\partial \dot{x}^l + q_l\, \partial/\partial \ddot{x}^l$$

with

$$p^l = \dot{\xi}^l - \dot{x}^l\tau_t \qquad \text{and} \qquad q^l = \ddot{\xi}^l - 2\ddot{x}^l\tau_t - \dot{x}^l\tau_{tt}.$$

In this case, the following theorem holds; the proof is the same.

*Theorem.* A necessary and sufficient condition that the divergence invariance of the integral functional $J(x^1, \ldots, x^n)$ and the conformal invariance of the Euler equations be equivalent is that the conformal factors $\alpha^l_k$ are given by

$$\alpha^l_k = -(\partial\xi^l/\partial x^k + \tau_t\delta^l_k)$$

where $\delta^l_k$ is the Kronecker delta.

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